Nonlinear Supersymmetry for Spectral Design in Quantum Mechanics

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Abstract.

Nonlinear (Polynomial, N-fold) SUSY approach to preparation of quantum systems with pre-planned spectral properties is reviewed. The full classification of ladder-reducible and irreducible chains of SUSY algebras in one-dimensional QM is given. Possible extensions of SUSY in one dimension are described. They include (no more than) $\mathcal{N} = 2$ extended SUSY with two nilpotent SUSY charges which generate the hidden symmetry acting as a central charge. Embedding stationary quantum systems into a non-stationary SUSY QM is shown to yield new insight on quantum orbits and on spectrum generating algebras.

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1. Introduction: Darboux intertwining, Schrödinger factorization, Witten SUSY mechanics in one basket

The concept of Supersymmetric Quantum Mechanics (SUSY QM) initially associated to the $0+1$ dimensional SUSY Field Theory [1] aimed at a simplified analysis of difficult problems of multi-dimensional QFT such as spontaneous SUSY breaking, vacuum properties beyond perturbation theory etc. [2, 3]. In addition, the realization of different SUSY algebras in Quantum Physics turned out to be easily achieved in certain QM models [4, 5].

Soon after its formulation SUSY QM was well identified with the Quantum Mechanics of isospectral systems described by Hamiltonians with almost coincident energy spectra [6]–[10]. SUSY manifested itself through intertwining Darboux [11] transformations between isospectral partners, used before in Mathematics [12]–[16]. The latter property presumably gave E. Schrödinger [17] a hint to factorize the corresponding Hamiltonians into a product of simplest, first-order Darboux operators [18]. Altogether the differential realization of SUSY in QM stimulated its application for the spectral design, *i.e.* for preparation of quantum potential systems with pre-planned energy spectra [19]–[39] and scattering data [9], [40]–[50] or potential profiles [51, 52] in a constructive way. For the spectral design-kit the non-linear supersymmetry in its differential realization [53]–[90] has become one of the most efficient tools to build various isospectral quantum systems with desired features. By now, several books and reviews [91]–[103] devoted to certain achievements and diverse applications of SUSY QM approach illuminated at length many of the above mentioned trends. In contradistinction, the role and the structure of nonlinear SUSY QM just developed in the last decade and has not been yet surveyed.

The interplay between the algebraic and differential properties of nonlinear supersymmetry, in the spectral design, is guiding our present work: we are going to clarify the benefits of algebraic SUSY approach to the old Darboux-Crum method, to develop, with these tools, the complete classification of differential realizations of non-linear SUSY algebras and to give insight to intrinsic links between SUSY isospectrality and hidden symmetries in particular quantum systems.

In what follows we restrict ourselves to the full analysis in the one-dimensional, one-component, stationary QM elaborating few important examples for the non-stationary Schrödinger equation and leave complex, matrix, multidimensional and/or relativistic (Dirac, Klein-Gordon etc.) equations without any detailed comments.

Let us start the retrospection of SUSY QM: consider two one-dimensional Schrödinger Hamiltonians $h^{\pm}$ defined on the line and assemble them into a matrix Super-Hamiltonian,

$$H = \begin{pmatrix} h^+ & 0 \\ 0 & h^- \end{pmatrix} = \begin{pmatrix} -\partial^2 + V_1(x) & 0 \\ 0 & -\partial^2 + V_2(x) \end{pmatrix}, \qquad \partial \equiv d/dx, \quad (1)$$

with non-singular real potentials. These Hamiltonians $h^{\pm}$ are supposed to have (almost) the same energy levels for bound states and/or the same spectral densities for continuum parts of spectra. Furthermore, assume that their *isospectral* connection is provided by intertwining with the Crum-Darboux [11, 12] transformation operators $q_N^{\pm}$,

$$h^+ q_N^+ = q_N^+ h^-, \quad q_N^- h^+ = h^- q_N^-, \tag{2}$$

where N-th order differential operators

$$q_N^{\pm} = \sum_{k=0}^{N} w_k^{\pm}(x)\partial^k, \quad w_N^{\pm} = const \equiv (\mp 1)^N. \tag{3}$$

The conventional, linear $\mathcal{N} = 1$ SUSY QM in the fermion number representation [2] is implemented by nilpotent supercharges $Q_1,\ Q_1^\dagger$ of first order in derivatives built from a real super-potential $\chi(x)$,

$$q_1^{\pm} \equiv \mp\partial + \chi(x); \qquad \Longrightarrow \qquad Q_1 = \begin{pmatrix} 0 & q_1^+ \\ 0 & 0 \end{pmatrix}, \quad Q_1^2 = \left(Q_1^\dagger\right)^2 = 0, \tag{4}$$

where † stands for the operation of hermitian conjugation (as well as we employ the differential operators with real coefficients the hermitian conjugation is equivalent to the operation of transposition, $Q^\dagger = Q^t$) .

The intertwining relations introduced in (2) result in the Supersymmetry for a Super-Hamiltonian H,

$$[H, Q_1] = [H, Q_1^\dagger] = 0. \tag{5}$$

The SUSY algebra is completed by the appropriate decomposition of the Super-Hamiltonian,

$$H = \{Q_1, Q_1^\dagger\} \iff h^+ = q_1^+ q_1^- = -\partial^2 + \chi^2 - \chi'; \quad h^- = q_1^- q_1^+ = -\partial^2 + \chi^2 + \chi', \tag{6}$$

which is in line with the Schrödinger one-step factorization [17, 18]. The notation $\chi' \equiv d\chi/dx$ has been employed.

At this stage, the super-potential χ is generated by zero-energy solutions of the Schrödinger equations (equivalently, zero-modes of supercharges $Q_1, Q_1^\dagger$),

$$h^{\pm}\phi_{(0)}^{\mp} = 0 = q_1^{\mp}\phi_{(0)}^{\mp}; \quad \phi_{(0)}^{-}(x) = \left(\phi_{(0)}^{+}(x)\right)^{-1} = \exp\left(-\int^x dy\chi(y)\right). \tag{7}$$

If $\chi(x)$ is a non-singular function then the zero-modes $\phi_{(0)}^{\mp}$ are nodeless. This leads to a non-negative physical spectrum in agreement with the SUSY algebra (6),

$$\langle\psi|H|\psi\rangle = \langle Q_1\psi|Q_1\psi\rangle + \langle Q_1^\dagger\psi|Q_1^\dagger\psi\rangle \geq 0, \tag{8}$$

for any L^2-normalizable, smooth wave function $\psi(x)$.

Several options exist for the choice of zero-modes of supercharges $Q_1, Q_1^\dagger$. If one of the zero-modes $\phi_{(0)}^{\mp}$ is normalizable then it becomes a ground state wave function of the Super-Hamiltonian H (*i.e.* of h^+ or h^-). But another one remains non-normalizable due to Eq.(7). Thus either q^- or q^+ deletes the ground state level of h^+ or h^-. When keeping in mind the spectral design program one can also interpret it conversely: if q^- deletes the lowest level of h^+ converting it into h^- then q^+ creates a new level for h^- transforming it into h^+.

Another option is realized by the non-normalizable nodeless functions $\phi^{\mp}_{(0)}$ when none of them belongs to the physical spectrum of Hamiltonians $h^{\pm}$. In this case the entire sets of physical eigenstates of the both Hamiltonians are put into the one-to-one correspondence by intertwining relations (2),

$$h^{\pm}\psi^{\pm}_{E} = E\psi^{\pm}_{E}; \quad E > 0; \quad \psi^{\mp}_{E} = \frac{1}{\sqrt{E}} q^{\mp}_{1}\psi^{\pm}_{E}, \tag{9}$$

and such Hamiltonians are strictly isospectral. In the SUSY vocabulary it is the case of "spontaneous" SUSY breaking as the lowest ground state of the Super-Hamiltonian H is degenerate.

The previous analysis has been based not only on the intertwining relations (2) but also on the factorization (6). However there is no such factorization for higher-order intertwining operators (3). What do we have for the latter ones instead?

2. From the ladder of SUSY's via Parasupersymmetry toward Polynomial SUSY

Let us proceed by recursion and discover different levels of isospectrality: from a simple Darboux transformation to a ladder or a dressing chain made of several simple Darboux steps. Actually the whole variety of elementary building blocks for spectral design can be well developed within the class of transformation operators $q^{\pm}_{2}$ of second-order in derivatives. One has to select the operators (3) with nonsingular coefficient functions which produce a nonsingular potential V_2 after intertwining (2) with the smooth initial potential V_1.

First of all, to produce the required transformation operators the two different linear SUSY systems may be "glued". Indeed, consider two Super-Hamiltonians $H_i,\ i = 1, 2$, Eq. (1), respectively two supercharges Q_i with super-potentials χ_i and supercharge components $r^{\pm}_{i} = \mp\partial + \chi_i$. Let us identify two elements of Super-Hamiltonians,

$$h^{-}_{1} = h^{+}_{2} + \lambda; \quad \chi^{2}_{1} + \chi'_{1} = \chi^{2}_{2} - \chi'_{2} + \lambda \tag{10}$$

with $E^{(0)}_{1} \geq \lambda \geq -E^{(0)}_{2}$ where $E^{(0)}_{1}$ and $E^{(0)}_{2}$ are ground state energies for h^{-}_{1} and h^{+}_{2} respectively. Evidently the constant shift of the Super-Hamiltonian $H_2 \to H_2 + \lambda$ does not break or change the supersymmetry.

After such a gluing the chain of intertwining relations (2) can be assembled into the supersymmetry transformation $[H_{ps}, Q_{ps}] = [H_{ps}, Q^{\dagger}_{ps}] = 0$ of the combined Super-Hamiltonian H_{ps} and the joint supercharges $Q_{ps}, Q^{\dagger}_{ps}$,

$$H_{ps} = \begin{pmatrix} h^{+}_{1} - \frac{\lambda}{2} & 0 & 0 \\ 0 & h^{-}_{1} - \frac{\lambda}{2} = h^{+}_{2} + \frac{\lambda}{2} & 0 \\ 0 & 0 & h^{-}_{2} + \frac{\lambda}{2} \end{pmatrix}; \quad Q_{ps} = \begin{pmatrix} 0 & r^{+}_{1} & 0 \\ 0 & 0 & r^{+}_{2} \\ 0 & 0 & 0 \end{pmatrix}, \tag{11}$$

where we have shifted both Super-Hamiltonians symmetrically to simplify the further algebra. However these supercharges $Q_{ps}, Q^{\dagger}_{ps}$ are not anymore nilpotent of order two and therefore do not mimic the Pauli principle. Still they are nilpotent of order three, $Q_{ps}^3 = \left(Q_{ps}^\dagger\right)^3 = 0$, and therefore the states carrying these charges obey the para-statistical principle. Thus we deal now with the Parasupersymmetry [104]–[109]. Furthermore the closure of superalgebra is not anymore given by Eq.(6). The lowest-order relation between the Super-Hamiltonian and the para-supercharges is trilinear,

$$Q_{ps}^\dagger Q_{ps}^2 + Q_{ps}^2 Q_{ps}^\dagger + Q_{ps} Q_{ps}^\dagger Q_{ps} = 2H_{ps} Q_{ps}. \tag{12}$$

This quantum system reveals triple degeneracy of levels with the possible exception for two lowest states. We draw also the reader's attention to a possibility to treat both intermediate Hamiltonians h_1^- and $h_2^+ + \lambda$ separately, in spite of their identification, thus doubling of the Hilbert spaces spanned by their eigenfunctions. It leads to a model of "Weak Supersymmetry" with quadruple level degeneracy [110] which may be susceptible to prolongation onto a weak-SUSY field theory.

Going back to the spectral design purposes, the para-supersymmetric dynamics contains redundant information, namely, about the intermediate Hamiltonian $h_1^- = h_2^+ + \lambda$. One is, in fact, interested in the final Hamiltonian h_2^- only as produced from the initial one, h_1^+ by means of a second-order Darboux transformation,

$$h_1^+ r_1^+ r_2^+ = r_1^+ h_1^- r_2^+ = r_1^+ (h_2^+ + \lambda) r_2^+ = r_1^+ r_2^+ (h_2^- + \lambda); \quad r_2^- r_1^- h_1^+ = (h_2^- + \lambda) r_2^- r_1^- . \tag{13}$$

Let us make a shortcut and define the two isospectral components $h^\pm$,

$$h^+ \equiv h_1^+ + \lambda_1 = r_1^+ r_1^- + \lambda_1; \quad h^- \equiv h_2^- + \lambda_2 = r_2^- r_2^+ + \lambda_2; \tag{14}$$

$$r_1^- r_1^+ + \lambda_1 = r_2^+ r_2^- + \lambda_2; \tag{15}$$

for the generalized Super-Hamiltonian (1) where we have employed a more general energy reference (shift by arbitrary $\lambda_{1,2}$). Evidently, $\lambda = \lambda_2 - \lambda_1$. Then the intertwining relations (2) are identical to Eq. (13) with $q_2^+ = r_1^+ r_2^+$ and the supersymmetry $[H, Q_2] = [H, Q_2^\dagger] = 0$ is generated by the conserved supercharges,

$$Q_2 = \begin{pmatrix} 0 & q_2^+ \\ 0 & 0 \end{pmatrix}, \quad Q_2^2 = \left(Q_2^\dagger\right)^2 = 0. \tag{16}$$

In place of Eq.(6), because of (14) the algebraic closure is given by,

$$\{Q_2, Q_2^\dagger\} = \begin{pmatrix} r_1^+ r_2^+ r_2^- r_1^- & 0 \\ 0 & r_2^- r_1^- r_1^+ r_2^+ \end{pmatrix} = (H - \lambda_1)(H - \lambda_2). \tag{17}$$

Thus we have obtained the second-order Polynomial SUSY algebra [53]-[60] as a concise form of isospectral deformation of a potential system accomplished by a ladder [6]–[10], [18]–[23] or a dressing chain [29]–[32] of a couple of one-step Darboux transformations or, equivalently, by a second-order Crum-Darboux intertwining [12, 27, 97] or by a blocking of two linear SUSY with partial overlapping of Super-Hamiltonians [53] ("weak SUSY" [110]), or by a tower of para-SUSY transformations [104]–[108].

In the modern SUSY vocabulary there are several synonyms for the higher-order SUSY algebra: originally it was named as a Polynomial (or Higher-derivative) one [53, 54], recently the title of N-fold SUSY has been suggested [73] and, at last, a more general term of Nonlinear SUSY has been used [80] with a certain reference to nonlinear

SUSY algebra arising in the conformal QM [111]. In what follows we will combine the first name and the last one depending on the structure of a superalgebra [85].

This Polynomial SUSY keeps track of essential spectral characteristics of the second-order SUSY (Crum-Darboux transformations) . Indeed, the zero-modes of intertwining operators $q_2^{\pm}$ or, equivalently, the zero-modes of the hermitian supercharges $Q_2^+ = Q_2 + Q_2^\dagger$; $Q_2^- = i(Q_2^\dagger - Q_2)$, form the basis of a finite-dimensional representation of the Super-Hamiltonian,

$$q_2^{\pm}\phi_i^{\pm}(x) = 0 = q_2^{\pm}h^{\mp}\phi_i^{\pm}(x); \quad i = 1, 2; \quad h^{\mp}\phi_i^{\pm}(x) = \sum_{j=1}^{2} S_{ij}^{\mp}\phi_j^{\pm}(x), \tag{18}$$

due to intertwining relations (2), (13). In terms of these Hamiltonian projections – constant matrices $\mathbf{S}^{\mp}$, the SUSY algebra closure takes the polynomial form [85] (see also [73]),

$$\left\{Q_2, Q_2^\dagger\right\} = \det\left[E\mathbf{I} - \mathbf{S}^+\right]_{E=H} = \det\left[E\mathbf{I} - \mathbf{S}^-\right]_{E=H} \equiv \mathcal{P}_2(H). \tag{19}$$

Thus both matrices $\mathbf{S}^{\mp}$ have the same set of eigenvalues which for the ladder construction (17) consists of λ_1, λ_2 . As the zero-mode set is not uniquely derived from (18) the matrices $\mathbf{S}^{\mp}$ are not necessarily diagonal. For instance, the equation $r_1^+ r_2^+ \phi^+(x) = 0$ has one zero-mode ϕ_2^+ obeying $r_2^+\phi_2^+(x) = 0$ and another one obeying $r_1^+\tilde{\phi}_1^+ = 0$; $\tilde{\phi}_1^+ = r_2^+\phi_1^+(x) \neq 0$. Evidently the zero-mode solution $\phi_1^+(x)$ is determined up to an arbitrary admixture of ϕ_2^+. When multiplying these linear equations by r_2^- one easily proves with the help of Eqs. (14),(15) that

$$(h^- - \lambda_2)\phi_2^+(x) = 0; \quad (h^- - \lambda_1)\phi_1^+(x) = C\phi_2^+(x); \quad \mathbf{S}^- = \begin{pmatrix} \lambda_1 & C \\ 0 & \lambda_2 \end{pmatrix}, \tag{20}$$

where C is an arbitrary real constant. If $\lambda_1 \neq \lambda_2$ then by the redefinition $(\lambda_1 - \lambda_2)\bar{\phi}_1^+ \equiv (\lambda_1 - \lambda_2)\phi_1^+ + C\phi_2^+(x)$ one arrives at the canonical diagonal form $\widetilde{\mathbf{S}}^-$. However in the *confluent case*, $\lambda_1 = \lambda_2 \equiv \lambda$, $C \neq 0$ it is impossible to diagonalize and by a proper normalization of the zero-mode ϕ_1^+ one gets another canonical form $\widetilde{\mathbf{S}}^-$ of this matrix – the elementary Jordan cell [112],

$$(h^- - \lambda)\phi_2^+(x) = 0; \quad (h^- - \lambda)\phi_1^+(x) = \phi_2^+(x); \quad \widetilde{\mathbf{S}}^- = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{21}$$

We display it to emphasize that in the confluent case the zero-mode ϕ_1^+ is not anymore a solution of the Schrödinger equation but it is a so-called adjoint solution [113] which can be obtained by differentiation, $\phi_1^+ = d\phi_2^+/d\lambda$. Yet the intermediate Hamiltonian $\tilde{h} = r_1^- r_1^+ + \lambda = r_2^+ r_2^- + \lambda$ is well defined and therefore the intermediate isospectral partner $\tilde{\phi}_1^+(x)$ of the zero-mode $\phi_1^+(x)$ is a solution of Schrödinger equation with the above Hamiltonian. The analysis of the matrix $\mathbf{S}^+$ is similar. Thus we have established that in general the Hamiltonian projection onto the subspace of (hermitian) supercharge zero-modes is not diagonalizable but can be always transformed into a canonical Jordan form.

To accomplish the description of Polynomial SUSY algebras generated by a second-order ladder one should take into consideration also the degenerate case when

$\lambda_1 = \lambda_2 \equiv \lambda$, $C = 0$. For this choice the matrix $\mathbf{S}^-$ is automatically diagonal and both zero modes $\phi^+_{1,2}(x)$ are (independent) solutions of the Schrödinger equation with the Hamiltonian h^-. Then it can be proved [85] that the intertwining operator q_2^+ is just a linear function of this Hamiltonian, $q_2^+ = \lambda - h^-$. Hence the intertwining is trivial $h^- = h^+$ and such supercharges must be eliminated. For higher-order SUSY the removal of such blocks in supercharges may lead to ladder irreducible SUSY algebras (see Section 4).

The very fact that the Hamiltonian is represented by finite matrices $\mathbf{S}^\pm$ is interpreted sometimes [78, 75] as a phenomenon of quasi-exact solvability (QES) [114, 115]: this point needs a certain comment. If one is seeking for some formal solutions of the Schrödinger equation, not necessarily normalizable and regular then such a formal QES can be accepted. But for the spectral design we impose physical boundary conditions and requirements of normalizability which are essential to define the energy levels properly. Then QES for physical wave functions is achieved only if one or both of eigenvalues belong to the energy spectrum of the Super-Hamiltonian. Obviously it is an exceptional situation which is not granted by the Polynomial SUSY itself.

Let us complete this section with the general description of the N-step ladder which entails the Polynomial Superalgebra of Nth-order. We introduce a set of first-order differential operators for intermediate intertwinings ,

$$r_l^\pm = \mp\partial + \chi_l(x), \quad l = 1, ..., N, \tag{22}$$

and the relevant number of intermediate super potentials $\chi_l(x)$. The set of the initial, $h^+ \equiv h_0$, the final, $h^- \equiv h_N$ and intermediate Hamiltonians, $h_l = -\partial^2 + v_l(x)$ consists of Schrödinger operators, so far nonsingular and real ones. They obey the ladder relations ("gluing"),

$$\begin{aligned} &h_l \equiv r_l^- \cdot r_l^+ + \lambda_l = r_{l+1}^+ \cdot r_{l+1}^- + \lambda_{l+1}, \quad l = 1, \dots, N-1, \\ &h_N \equiv h^- = r_N^- \cdot r_N^+ + \lambda_N, \quad h_0 \equiv h^+ = r_1^+ \cdot r_1^- + \lambda_1. \end{aligned} \tag{23}$$

These gluing relations are provided by the (dressing) chain equations on super-potentials,

$$v_l(x) = (\chi_l(x))^2 + (\chi_l(x))' + \lambda_l = (\chi_{l+1}(x))^2 - (\chi_{l+1}(x))' + \lambda_{l+1} \tag{24}$$

The corresponding intertwining (Darboux) transformations hold for each adjacent pair of Hamiltonians,

$$h_{l-1} \cdot r_l^+ = r_l^+ \cdot h_l, \quad r_l^- \cdot h_{l-1} = h_l \cdot r_l^-, \tag{25}$$

and therefore the chain of N overlapping SUSY systems is properly built,

$$\begin{aligned} &H_l = \begin{pmatrix} h_{l-1} & 0 \\ 0 & h_l \end{pmatrix}, \qquad R_l = \begin{pmatrix} 0 & r_l^+ \\ 0 & 0 \end{pmatrix}; \\ &[H_l, R_l] = [H_l, R_l^\dagger] = 0, \qquad H_l - \lambda_l = \{R_l, R_l^\dagger\}, \end{aligned} \tag{26}$$

This Chain Supersymmetry can be equally converted into a N-order Parasupersymmetry similar to Eqs. (11), (12) which however we do not need for the further construction.

Now let us disregard a chain of intermediate Hamiltonians between h^+ and h^- and proceed to the Higher-derivative $\simeq$ Polynomial $\simeq$ Nonlinear SUSY algebra for the Super-Hamiltonian H given in Eq.(1). The intertwining between h^+ and h^- is realized by the Crum-Darboux operators,

$$q_N^+ = r_1^+ \dots r_N^+, \quad q_N^- = r_N^- \dots r_1^- . \tag{27}$$

The SUSY symmetry $[H, Q_N] = [H, Q_N^\dagger] = 0$, is still performed by the supercharges of the same matrix structure (16) and the Super-Hamiltonian is represented by finite-dimensional matrices on the subspaces of supercharge zero-modes,

$$q_N^\pm \phi_i^\pm(x) = 0; \quad i = 1, 2, \dots, N; \quad h^\mp \phi_i^\pm(x) = \sum_{j=1}^N S_{ij}^\mp \phi_j^\pm(x), \tag{28}$$

due to intertwining relations (2). In terms of the constant matrices $\mathbf{S}^\mp$, the algebraic closure is just represented by a non-linear SUSY relation [85, 73],

$$\left\{Q_N, Q_N^\dagger\right\} = \det\left[E\mathbf{I} - \mathbf{S}^+\right]_{E=H} = \det\left[E\mathbf{I} - \mathbf{S}^-\right]_{E=H} \equiv \mathcal{P}_N(H) = \prod_{l=1}^N (H - \lambda_l) . \tag{29}$$

The disposition of the real roots λ_l against the energy levels of the Super-Hamiltonian is assumed to provide the positivity of the superalgebra (29) (see next Section). Namely if m lowest energy levels, $\{E_j\}$, $j = 0, \dots, m-1$ are among the roots, $\{E_j\} \subset \{\lambda_l\}$ then all $\lambda_l < E_m$. Again both matrices $\mathbf{S}^\mp$ have the same set of eigenvalues which for the ladder construction (23) consists of $\lambda_1, \dots \lambda_N$. If the degenerate roots appear then the canonical forms $\widetilde{\mathbf{S}}^\mp$ of the latter matrices are not necessarily diagonal and may consist of Jordan cells. If all intermediate h_l are hermitian, nonsingular and superpotentials are taken real, then λ_l are real and each ladder step is well defined. What will happen if we extend the class of Polynomial SUSY algebras admitting complex λ_l and singular h_l ?

3. Algebraic classification of Polynomial SUSY QM and its functional realization: irreducibility of type I, II, and III

Let us now examine which circumstances may obstruct the SUSY ladder decomposition of a Polynomial SUSY algebra. In fact, all elementary "bricks" irreducible to a chain of one-step Darboux intertwinings are well revealed for the second-order SUSY algebra described in the previous section.

For a supercharge of second order in derivatives with real coefficient functions one can find real zero-modes of the intertwining operators $q_2^\pm$ and, further on, the 2×2 matrix representation (18) for the Super-Hamiltonian components $h^\pm$ by matrices $\mathbf{S}^\pm$. The latter matrices are real but, in general, not symmetric. Therefore the first obstruction for the ladder decomposition may arise because the reduction to a Jordan form has not given real eigenvalues. For instance, if $h^+\phi_i^-(x) = \bar{\lambda}\epsilon_{ik}\phi_i^-(x)$ then the eigenvalues of $\mathbf{S}^+ = \bar{\lambda}\hat{\epsilon}$ are imaginary, mutually conjugated $\pm i\bar{\lambda}$. The possibility of complex pairs of mutually conjugated roots in a Polynomial SUSY algebra can be easily read off from its closure (19) as for supercharges with real coefficients polynomials $\mathcal{P}_2(H)$ possess real coefficients. We call this kind of irreducibility to be of type I. Its elementary block corresponds to the polynomial $\mathcal{P}_2^{(I)}(H) = (H+a)^2 + d, \quad d > 0$ and its analytical properties have been investigated in [54], [84]. Some examples of related isospectral potentials are described in [82].

Next, one has to ensure the positivity of the SUSY algebra relation (19) in a particular differential realization of a Super-Hamiltonian H with real non-singular potentials and the supercharges $Q_N, Q_N^\dagger$ (with $N = 2$ in our case) made of differential operators with real coefficients. Let the energy spectrum $E_j; j = 0, 1, \ldots; E_j < E_{j+1}$ of H be discrete, for simplicity. Then,

$$\mathcal{P}_N(E_j) = \langle Q_N \Psi_j | Q_N \Psi_j \rangle + \langle Q_N^\dagger \Psi_j | Q_N^\dagger \Psi_j \rangle \geq 0, \tag{30}$$

if the action of supercharges is well defined in the Hilbert space spanned by eigenfunctions of a Super-Hamiltonian. It can be easily extended on a continuum energy spectrum as well using wave packets.

Thus for regular potentials the allowed disposition of polynomial roots $\Longleftrightarrow$ zero-modes of a supercharge — provides non-negative values of $\mathcal{P}(E)$ for *each* energy level of a Hamiltonian. To be definite, one may have the following cases for the allocation of polynomial real roots (for a pair of complex, mutually conjugated roots the positivity is obvious) .

Case A. $\lambda_1 \leq \lambda_2 \leq E_0$ or $\lambda_1 = E_0$; $\lambda_2 = E_1$.
The related SUSY algebra has a chain or ladder realization. In other words, it is reducible, in principle, because one gradually can add/remove λ_1 and then λ_2 without breaking the positivity of intermediate SUSY algebra. The coincidence of roots and energies correspond in the spectral design to deleting/inserting energy levels. For instance, if $\lambda_1 = E_0$; $\lambda_2 = E_1$ then two pairs of zero-modes of $q_2^\pm$ can be chosen as solutions of two Schrödinger equations with Hamiltonians $h^\mp$. Repeating the arguments of Section 1 one can conclude [20] that the energy levels E_0, E_1 may well appear in any of the Hamiltonians $h^\pm$ but each level only once, either in h^+ or in h^-. Thus the entire variety of spectral design tools happens to be at our disposal: namely, one may delete two lowest levels, replace the ground-state level by a different one and add two more levels below the ground-state one.

Case B. $E_0 < \lambda_1 < \lambda_2 \leq E_1$ or $E_j \leq \lambda_1 < \lambda_2 \leq E_{j+1}$, $1 \leq j$.
A pair of real roots is placed between adjacent energy levels. If one of the solutions with eigenvalues $\lambda_{1,2}$ (zero-modes of the supercharges) is normalizable we perform the insertion/deletion of an excited energy state. Thus with these means one can delete two adjacent excited levels, shift the position of an excited level and add two more excited levels between two adjacent ones. Evidently, such an algebra cannot be safely decomposed into a chain of two linear SUSY as the removal of any of roots $\lambda_{1,2}$ immediately breaks the positivity in (30). Then the intermediate Hamiltonian acquires inevitably a real but singular potential leading to the loss of isospectrality. The related Darboux transformations had been known in 50ties [14]. We call this irreducibility to be of type II. Examples and certain theorems are given in [61], [82].

Case C. $E_j < \lambda_1 = \lambda_2 \leq E_{j+1}$, $0 \leq j$.
This is a confluent case which seems to arise as a limit of the previous one. However, let us remind that the one-dimensional QM does not allow degenerate levels. Besides, let's assume that the matrix representation for the corresponding Super-Hamiltonian contains a non-trivial Jordan cell. Then the limit becomes quite delicate as one of the zero-modes is not a solution of the Schrödinger equation but represents an adjoint function [113] (see Eq. (21) and discussion afterwards). This is why we specify this case as a separate one named to be of type III. With such an intertwining operation one may insert/delete odd number of excited levels in an economical way. One may find more information on the analytical properties of related potentials in [87].

One can apply these second-order blocks and build an Nth-order Polynomial SUSY system. Their general form is again given by Eq. (29) allowing the presence of complex conjugated roots λ_l. Let us rewrite it taking into account the possibility of complex and degenerate roots,

$$\left\{Q_N, Q_N^\dagger\right\} = \mathcal{P}_N(H) = \prod_{l=1}^{n} (H - \lambda_l)^{\nu_l} \prod_{j=1}^{m} [(H + a_j)^2 + d_j]^{\mu_j},$$
$$N = \sum_{l=1}^{n} \nu_l + 2\sum_{j=1}^{m} \mu_j, \quad d_j > 0. \tag{31}$$

We stress that in the general case the Hamiltonian projections onto the zero-mode spaces of intertwining operators $q_N^\pm$ are finite $N \times N$ matrices $\mathbf{S}^\mp$ and the Polynomial SUSY algebra can be represented by Eqs. (29). Inequality (30) is certainly valid for $\mathcal{P}_N(H)$.

Irreducible elements of type II are not straightforwardly seen in the structure of the Polynomial SUSY algebra and can be unraveled only after the inspection of disposition of polynomial roots in respect to energy levels. They fill the chain of intertwining operators being even order in derivatives and placing a pair (or few pairs) of real roots $\lambda_l < \cdots < \lambda_{l+2k-1}$ (supercharge zero-modes) between two successive energy levels $E_j < \lambda_l < \cdots < \lambda_{l+2k-1} < E_{j+r}$; $r - 1 \leq 2k$ if the intermediate levels $\{E_{j+1}, \ldots, E_{j+r-1}\} \subset \{\lambda_l, \ldots, \lambda_{l+2k-1}\}$; $r - 1 \leq 2k$. The eigenvalues E_j, E_{j+r} are assumed not to coincide with any of polynomial roots. Then the polynomial

$$\mathcal{P}^{(\mathrm{II})}_{2k,j,r}(H) = \prod_{i=0}^{2k-1} (H - \lambda_{l+i}); \quad \mathcal{P}^{(\mathrm{II})}_{2k,j,r}(E_m) > 0. \tag{32}$$

When the related zero-modes coincide with some eigenfunctions of the Super-Hamiltonian the pertinent supercharges create or annihilate particular excited states in the components $h^\pm$ of the Super-Hamiltonian.

Irreducible elements of type III fill the chain of Darboux transformations being represented by even-order intertwining operators responsible for allocation of even number of real confluent roots $\lambda_l = \lambda_{l+1} = \cdots = \lambda_{l+2\nu_l-1}$ (= supercharge zero-modes) between two adjacent energy levels $E_j < \lambda_l \leq E_{j+1}$ for some $0 \leq j$,

$$\mathcal{P}^{(\mathrm{III})}_{2\nu_l,j}(H) = (H - \lambda_l)^{2\nu_l}; \quad \mathcal{P}^{(\mathrm{III})}_{2\nu_l,j}(E_m) > 0. \tag{33}$$

No more than two zero-modes may be solutions of the Super-Schrödinger equation, in particular, eigenfunctions of the Super-Hamiltonian if $E_j = \lambda_l$. Other zero-modes are adjoint functions [113] to a solution of the Schrödinger equation.

Finally, in general, the polynomial in Eq. (31) can be factorized into the product of the elements (32) and (33) with roots located between two successive or adjacent levels. The related ladder of Darboux transformations consists of reducible steps as well as of few irreducible elements of type I, II, III displayed in Eqs. (31), (32), (33).

Yet the open question remains whether any irreducible element of type II or III ((32) or (33)) can decomposed into the ladder of second-order irreducible blocks with regular hermitian intermediate Hamiltonians between them in the ladder. We are informed that essential progress in this direction has been made by A. V. Sokolov and hope to see it published soon.

On the other hand, an experienced SUSY designer may be somewhat puzzled with the very existence of irreducible super-transformations. Indeed it is quite conceivable that a pair of supercharge zero-modes or even a pair of new excited energy levels of the Super-Hamiltonian can be inserted by successive application of first-order intertwining (super) transformations between regular Hamiltonians following the ladder algorithm described in the previous Section. But the order of the relevant ladder of first-order transformations and respectively of the final Polynomial SUSY will be evidently higher than two. We come to the problem of possible relationship between first-order reducible and irreducible SUSY algebras having the same Super-Hamiltonian.

The related important question concerns the degenerate roots. By general arguments these roots are distributed between different Jordan cells in the canonical forms $\widetilde{\mathbf{S}}^{\pm}$ of the matrices $\mathbf{S}^{\pm}$. One can inquire on how many Jordan cells may coexist and if several cells appear then what is their role in the supercharge structure. All these problems are clarified with the help of the Strip-off theorem [85].

4. From reducible SUSY to irreducible one when equipped by the Strip-off theorem

Let first elucidate the possible redundancy in supercharges which can be eliminated without any changes in the Super-Hamiltonian (*i.e.* preserving the same potentials). There exists a trivial possibility when the intertwining operators $q_N^{\pm}$ and $p_{N_1}^{\pm}$ for $N > N_1$ are related by a polynomial factor $F(x)$ depending on the Hamiltonian,

$$q_N^{\pm} = F(h^{\pm}) p_{N_1}^{\pm} = p_{N_1}^{\pm} F(h^{\mp}). \tag{34}$$

Obviously in this case the appearance of the second supercharge does not result in any new restrictions on potentials.

Thus the problem arises of how to separate the nontrivial part of a supercharge and avoid numerous SUSY algebras generated by means of "dressing" (34). It can be systematically realized with the help of the following

"Strip-off" theorem.

Let's admit the construction given by Eqs. (28) and (29). Then
a) the matrix $\mathbf{S}^-$ (or $\mathbf{S}^+$) generated by the Hamiltonians h^- (or h^+) on the subspace of zero-modes of the operator q_N^+ (or q_N^-), after reduction into the Jordan form $\widetilde{\mathbf{S}}^-$ (or $\widetilde{\mathbf{S}}^+$), may contain only one or two Jordan cells with equal eigenvalues λ_l;
b) assume that there are n pairs (and no more) of Jordan cells with equal eigenvalues and with the sizes ν_l of a smallest cell in the l-th pair; then this condition is necessary and sufficient to ensure for the intertwining operator q_N^+ (or q_N^-) to be represented in the factorized form:

$$q_N^\pm = p_{N_1}^\pm \prod_{l=1}^{n} (\lambda_l - h^\mp)^{\nu_l}, \tag{35}$$

where $p_{N_1}^\pm$ are intertwining operators of order $N_1 = N - 2\sum_{l=1}^n \nu_l$ which cannot be decomposed further on in the product similar to (34) with $F(x) \neq const.$

Remark. The matrices $\widetilde{\mathbf{S}}^\pm$ cannot contain more than two Jordan cells with the same eigenvalue λ because otherwise the operator $\lambda - h^\pm$ would have more than two linearly independent zero-modes (not necessarily normalizable).
The full proof of this theorem has been performed in [85].

Let us illustrate the Theorem in the Example:
the matrix $\mathbf{S}^-$ for the intertwining operator q_3^+ with Jordan cells of different size having the same eigenvalue. It is generated by the operators,

$$p^\pm = \mp\partial + \chi, \quad h^\pm = p^\pm p^\mp + \lambda, \quad q_3^+ = -p^+ p^- p^+ = p^+(\lambda - h^-). \tag{36}$$

Respectively:

$$\begin{array}{lccc} \phi_1^+ : & p^- p^+ \phi_1^+ = \phi_2^+ & \longrightarrow & h^- \phi_1^+ = \lambda \phi_1^+ + \phi_2^+; \\ \phi_2^+ : & p^+ \phi_2^+ = 0 & \longrightarrow & h^- \phi_2^+ = \lambda \phi_2^+; \\ \phi_3^+ : & p^+ \phi_3^+ \neq 0,\, p^- p^+ \phi_3^+ = 0 & \longrightarrow & h^- \phi_3^+ = \lambda \phi_3^+; \end{array} \qquad \mathbf{S}^- = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}. \tag{37}$$

As a consequence of the "Strip-off" Theorem one finds that the supercharge components cannot be factorized in the form (34) if the polynomial $\tilde{\mathcal{P}}_N(x)$ in the SUSY algebra closure (29) does not have the degenerate zeroes. Indeed the SUSY algebra closure contains the square of polynomial $F(x)$, for instance,

$$q_N^- q_N^+ = F(h^-) p_{N_1}^- p_{N_1}^+ F(h^-) = F^2(h^-) \tilde{\mathcal{P}}_{N_1}(h^-), \tag{38}$$

where $\tilde{\mathcal{P}}_{N_1}(x)$ is a polynomial of lower order, $N_1 \leq N - 2$. Therefore each zero of the polynomial $F(x)$ will produce a double zero in the SUSY algebra provided by (38).

Thus the absence of degenerate zeroes is sufficient to have supercharges without redundancy in the sense of Eq. (34). However it is not necessary because the degenerate zeroes may well arise in the confluent ladder construction giving new pairs of isospectral potentials [54].

Now we proceed to uncover the origin of irreducible, type-II and -III transformations based on the strip-off factorization. For clarity let us consider an example of irreducible SUSY of type II with supercharges of second order in derivatives (see previous Section).

Suppose that it realizes insertion of two new energy levels between the ground and first excited states. Then three lowest energy levels $E_0 < E_1 < E_2$ are of importance to study the relevant SUSY systems: the ground state level is degenerate between SUSY partners h^+ and h^-, *i.e.* $E_0^+ = E_0^-$ whereas the two excited levels are present only in the spectrum of h^-.

One can use the ladder construction (23)–(27) to prepare the same spectral pattern. For this purpose, intertwining operators (27) of, at least, fourth order in derivatives must be employed. Indeed, one can prescribe the ladder steps for $q_4^\pm$ as follows: start from a pair of isospectral Hamiltonians with ground state energies E_3; generate the level E_0 in the Hamiltonian h^+ using the intertwining operators $r_1^\pm$, then sequentially create in the spectrum of h^- the state with energy $E_2 < E_3$ by means of $r_2^\pm$, next the energy level $E_1 < E_2$ using $r_3^\pm$ and finally the ground state with energy $E_0 < E_1$ exploiting $r_4^\pm$. These elementary steps are reflected in zero-modes of the intertwining operators $q_4^+ = r_1^+ r_2^+ r_3^+ r_4^+$ and $q_4^- = r_4^- r_3^- r_2^- r_1^-$. Namely the ground state of h^+ is a zero-mode of r_1^- (*i.e.* of q_4^-) whereas the eigenstates of h^- corresponding to E_0, E_1, E_2 are annihilated by the product $r_2^+ r_3^+ r_4^+$ (*i.e.* by q_4^+) according to Eq. (28). In particular, the ground state of h^- is a zero-mode of r_4^+. As the ground state energies coincide for $h^\pm$ the Hamiltonian projections on the $q_4^\pm$ zero-mode space – the matrices $\mathbf{S}^\pm$ are, in general, not diagonalizable but have one rank-two Jordan cell each. Thus, for instance,

$$\mathbf{S}^- = \begin{pmatrix} E_0 & 0 & 0 & C \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_1 & 0 \\ 0 & 0 & 0 & E_0 \end{pmatrix} \quad \Longrightarrow \quad \tilde{\mathbf{S}}^- = \begin{pmatrix} E_0 & C & 0 & 0 \\ 0 & E_0 & 0 & 0 \\ 0 & 0 & E_2 & 0 \\ 0 & 0 & 0 & E_1 \end{pmatrix}, \tag{39}$$

where a non-zero constant C can be normalized to $C = 1$. The canonical Jordan form $\tilde{\mathbf{S}}^-$ in (39) is achieved by means of re-factorization $q_4^+ = r_1^+ r_2^+ r_3^+ r_4^+ = r_1^+ \tilde{r}_2^+ \tilde{r}_3^+ \tilde{r}_4^+$ so that the annihilation of ground state for h^- is associated now with $\tilde{r}_2^+$. Respectively, the Polynomial SUSY algebra shows up one degenerate root,

$$\mathcal{P}_4(H) = (H - E_0)^2 (H - E_1)(H - E_2). \tag{40}$$

The Strip-off theorem tells us that this fourth-order algebra cannot be optimized to a lower-order one because there is no replication of roots in different Jordan cells of $\tilde{\mathbf{S}}^\pm$ matrices. However one may perform fine-tuning of Darboux transformation parameters to provide the constant $C = 0$ in (39). This peculiar choice moves the SUSY system into the environment of the Strip-off theorem as now two rank-one cells in (39) contain the same eigenvalue E_0 . The SUSY algebra is still given by Eq. (40) but the intertwining operators reveal a redundancy,

$$q_4^+ = (E_0 - h^-)\, q_2^+. \tag{41}$$

By construction the left-hand side of this relation is fully factorizable in elementary binomials r_j^+ with hermitian nonsingular intermediate Hamiltonians. But in the right-hand side the operator $q_2^+ = \tilde{r}_3^+ \tilde{r}_4^+$ does not admit a further factorization with a nonsingular intermediate Hamiltonian because after removal of the redundant factor

$(h^- - E_0)$ such a factorization is forbidden by the positivity of the SUSY algebra, Eq. (30).

One can easily extrapolate the previous argumentation to the case of additional degeneracy of excited levels $E_1 = E_2$ to analyze the irreducible SUSY of type III. Thus we reach the important conclusions that:
a) the factorization (27) of intertwining operators $q_N^\pm$ is not unique and there exist options to have more reducible ladders and less reducible ones with a larger number of singular intermediate Hamiltonians;
b) (some of) irreducible algebras of type-II and -III can be identified with special cases of fully reducible ladder-type algebras when the Hamiltonian projections $\mathbf{S}^\pm$ have an appropriate number of pairs of Jordan cells with coinciding eigenvalues;
c) there are many (almost) isospectral systems with different pattern of excited states which cannot be interrelated with the help of irreducible Darboux transformations of type-II or -III but can be built with the help of higher-order reducible SUSY ladder.

Yet one may substantially gain effectiveness when the spectral design program allows to apply the irreducible transformations of type-II or -III in order to embed a couple of energy levels between two excited ones. Thus a more rigorous investigation of the relationship between the reducible and irreducible intertwinings is welcome. Especially important is the proof that any type-II, -III irreducible SUSY can be embedded (for the same Super-Hamiltonian) into a reducible ladder SUSY.

5. More supercharges $\Longleftrightarrow$ Extended SUSY $\Longleftrightarrow$ Hidden symmetry

The possibility of two supercharges for a quantum SUSY system was mentioned in [116] (see the preprint version). Namely the conserved supercharges $Q, Q^\dagger$ with complex coefficient functions in intertwining components $q_N^\pm$ accounted for two SUSY algebras for a hermitian Super-Hamiltonian H: one for their "real" parts $K, K^\dagger$ and another one for their "imaginary" parts $P, P^\dagger$ where real and imaginary parts are referred to coefficients in the intertwining operators $q_N^\pm = k_N^\pm + ip_{N_1}^\pm$.

Let us examine the general possibility to have several supercharges for the same Super-Hamiltonian. First we remind that a number of supercharges can be produced with the help of multiplication by a polynomial of the Hamiltonian (see Section 4). Certainly such supersymmetries are absolutely equivalent for the purposes of spectral design and one must get rid of them. As shown in [85] one can always optimize the infinite set of possible supercharges so that no more than two nontrivial supercharges remain which are used to generate all other ones by "dressing" with polynomials of the Hamiltonians. Thus in one-dimensional QM one may have the $\mathcal{N} = 1, 2$ SUSY only.

Correspondingly we consider now a general case when the Super-Hamiltonian H admits two supersymmetries with supercharges K and P generated by differential intertwining operators of order N and N_1 respectively,

$$[H, K] = [H, P] = [H, K^\dagger] = [H, P^\dagger] = 0. \tag{42}$$

The second supercharge P is assumed to be a differential operator of lower order $N_1 < N$.

To close the algebra one has to include all anti-commutators between supercharges, *i.e.* the full algebra based on two supercharges K and P with real intertwining operators. Two supercharges generate two Polynomial SUSY,

$$\left\{K, K^\dagger\right\} = \tilde{\mathcal{P}}_N(H), \quad \left\{P, P^\dagger\right\} = \tilde{\mathcal{P}}_{N_1}(H). \tag{43}$$

The closure of the extended, $\mathcal{N} = 2$ SUSY algebra is given by

$$\begin{aligned} \left\{P, K^\dagger\right\} &\equiv \mathcal{R} = \begin{pmatrix} p^+_{N_1} k^-_N & 0 \\ 0 & k^-_N p^+_{N_1} \end{pmatrix}, \\ \left\{K, P^\dagger\right\} &\equiv \mathcal{R}^\dagger = \begin{pmatrix} k^+_N p^-_{N_1} & 0 \\ 0 & p^-_{N_1} k^+_N \end{pmatrix}. \end{aligned} \tag{44}$$

Apparently the components of operators $\mathcal{R}$, $\mathcal{R}^\dagger$ are differential operators of $N+N_1$ order commuting with the Hamiltonians $h^\pm$, hence $\mathcal{R}$, $\mathcal{R}^\dagger$ are symmetry operators for the Super-Hamiltonian. However, in general, they are not polynomials of the Hamiltonians $h^\pm$ and these symmetries impose certain constraints on potentials.

All four operators $\tilde{\mathcal{P}}_N(H)$, $\tilde{\mathcal{P}}_{N_1}(H)$, $\mathcal{R}$, $\mathcal{R}^\dagger$ mutually commute. Moreover the hermitian matrix describing this $\mathcal{N} = 2$ SUSY,

$$\mathcal{Z}(H) = \begin{pmatrix} \tilde{\mathcal{P}}_N(H) & \mathcal{R} \\ \mathcal{R}^\dagger & \tilde{\mathcal{P}}_{N_1}(H) \end{pmatrix}, \quad \det\left[\mathcal{Z}(H)\right] = \tilde{\mathcal{P}}_N \tilde{\mathcal{P}}_{N_1} - \mathcal{R}\mathcal{R}^\dagger = 0, \tag{45}$$

is degenerate. Therefore it seems that the two supercharges are not independent and by their redefinition (unitary rotation) one might reduce the extended SUSY to an ordinary $\mathcal{N} = 1$ one. However such rotations cannot be global and must use non-polynomial, pseudo-differential operators as “parameters”. Indeed, the operator components of the “central charge” matrix $\mathcal{Z}(H)$ have different order in derivatives. Thus, globally the extended nonlinear SUSY deals with two sets of supercharges but when they act on a given eigenfunction of the Super-Hamiltonian H one could, in principle, perform the energy-dependent rotation and eliminate a pair of supercharges. Nevertheless this reduction can be possible only after the constraints on potentials have been resolved.

Let us find the formal relation between the symmetry operators $\mathcal{R}$, $\mathcal{R}^\dagger$ and the Super-Hamiltonian. These operators can be decomposed into a hermitian and an anti-hermitian parts,

$$\mathcal{B} \equiv \frac{1}{2}(\mathcal{R} + \mathcal{R})^\dagger \equiv \begin{pmatrix} b^+ & 0 \\ 0 & b^- \end{pmatrix}, \quad i\mathcal{E} \equiv \frac{1}{2}(\mathcal{R} - \mathcal{R}^\dagger) \equiv i \begin{pmatrix} e^+ & 0 \\ 0 & e^- \end{pmatrix}. \tag{46}$$

The operator $\mathcal{B}$ is a differential operator of even order and therefore it may be a polynomial of the Super-Hamiltonian H. But if the operator $\mathcal{E}$ does not vanish identically it is a differential operator of *odd* order and cannot be realized by a polynomial of H.

The first operator plays essential role in the one-parameter non-uniqueness of the SUSY algebra. Indeed, one can always redefine the higher-order supercharge as follows,

$$K^{(\zeta)} = K + \zeta P, \quad \left\{K^{(\zeta)}, K^{(\zeta)\dagger}\right\} = \tilde{\mathcal{P}}^{(\zeta)}_N(H), \tag{47}$$

keeping the same order N of Polynomial SUSY for arbitrary real parameter ζ. From (47) one gets,

$$2\zeta\mathcal{B}(H) = \tilde{\mathcal{P}}_N^{(\zeta)}(H) - \tilde{\mathcal{P}}_N(H) - \zeta^2\tilde{\mathcal{P}}_{N_1}(H), \tag{48}$$

thereby the hermitian operator $\mathcal{B}$ is a polynomial of the Super-Hamiltonian of the order $N_b \leq N-1$. Let's use it to unravel the Super-Hamiltonian content of the operator $\mathcal{E}$,

$$\mathcal{E}^2(H) = \tilde{\mathcal{P}}_N(H)\tilde{\mathcal{P}}_{N_1}(H) - \mathcal{B}^2(H), \tag{49}$$

which follows directly from (45) and (46). As the (nontrivial) operator $\mathcal{E}(H)$ is a differential operator of odd order N_e it may have only a realization non-polynomial in H being a square root of (49) in an operator sense. This operator is certainly non-trivial if the sum of orders $N+N_1$ of the operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ is odd and therefore $N_e = N+N_1$. The opposite statement was also shown in [85], namely if the symmetry operator is non-zero then for any choice of the operators $k_N^{\pm}$ and $p_{N_1}^{\pm}$ an optimal set of independent supercharges (possibly of lower orders) can be obtained which is characterized by an odd sum of their orders.

The existence of a nontrivial symmetry operator $\mathcal{E}$ commuting with the Super-Hamiltonian results in common eigenstates which however are not necessarily physical wave functions. In general they may be combinations of two solutions of the Shrödinger equation with a given energy, the physical and unphysical ones. But if the symmetry operator $\mathcal{E}$ is hermitian in respect to the scalar product of the Hilbert space spanned by the eigenfunctions of the Super-Hamiltonian H then both operators have a common set of physical wave functions. This fact imposes quite rigid conditions on potentials.

In particular, for intertwining operators with sufficiently smooth coefficient functions having constant asymptotics for $x \longrightarrow \pm\infty$ the symmetry operator $\mathcal{E}$ has the similar properties and is evidently hermitian. In this case one has non-singular potentials with constant asymptotics and therefore a continuum energy spectrum of H with wave functions satisfying the scattering conditions. Thus the incoming and outgoing states, $\psi_{in}(x)$ and $\psi_{out}(x)$, at large x are conventionally represented by combinations of plane waves which are solutions of the Schrödinger equation for a free particle,

$$\begin{aligned} \psi(x)|_{x\to-\infty} &\longrightarrow \exp(ik_{in}x) + R(k_{in})\exp(-ik_{in}x), \\ \psi(x)|_{x\to+\infty} &\longrightarrow (1+T(k_{out}))\exp(ik_{out}x), \end{aligned} \tag{50}$$

where the reflection, $R(k_{in})$, and transmission, $T(k_{out})$, coefficients are introduced. Since the symmetry is described by a differential operator of odd order which tends to an antisymmetric operator with constant coefficients the eigenstates of this operator approach asymptotically individual plane waves $\sim \exp(\pm ikx)$ with opposite eigenvalues $\sim \pm kf(k^2)$ and cannot be superimposed. Hence the eigenstate of the Super-Hamiltonian with a given value of the operator $\mathcal{E}$ may characterize only the transmission and cannot have any reflection, $R(k_{in}) = 0$. We conclude that the corresponding partner potentials $V_{1,2}$ inevitably belong to the class of transparent or reflectionless ones [117]. Such a symmetry may have relation to the Lax method in the soliton theory [16].

As the symmetry operator $\mathcal{E}$ is hermitian its eigenvalues are real but, by construction, its coefficients are purely imaginary. Since the wave functions of bound states of the system H can be always chosen real we conclude that they must be zero-modes of the operator $\mathcal{E}(H)$,

$$\mathcal{E}(H)\psi_i = \mathcal{E}(E_i)\psi_i = 0, \quad \tilde{\mathcal{P}}_N(E_i)\tilde{\mathcal{P}}_{N_1}(E_i) - \mathcal{B}^2(E_i) = 0, \tag{51}$$

which represents the algebraic equation on bound state energies of a system possessing two supersymmetries. Among solutions of (51) one reveals also a zero-energy state at the bottom of continuum spectrum. On the other hand one could find also the solutions which are not associated to any bound state. The very appearance of such unphysical solutions is accounted for by the trivial possibility to replicate supercharges by their multiplication by the polynomials of the Super-Hamiltonian as discussed in Sec. 4.

6. A simple but useful example of Extended SUSY

Let us examine the algebraic structure of the simplest non-linear SUSY with two supercharges,

$$\begin{aligned} k^{\pm} &\equiv \partial^2 \mp 2f(x)\partial + \tilde{b}(x) \mp f'(x); \\ p^{\pm} &\equiv \mp\partial + \chi(x), \end{aligned} \tag{52}$$

induced by the complex supercharge of second order in derivatives [85, 116]. The supersymmetries (42) generated by K, $K^\dagger$ and P, $P^\dagger$ prescribe that

$$\begin{aligned} V_{1,2} &= \chi^2 \mp \chi' = \mp 2f' + f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a, \\ \tilde{b} &= f^2 - \frac{f''}{2f} + \left(\frac{f'}{2f}\right)^2 + \frac{d}{4f^2}, \end{aligned} \tag{53}$$

where χ, f are real functions and a, d are real constants. The related superalgebra closure for K, $K^\dagger$ and P, $P^\dagger$ takes the form,

$$\{K, K^\dagger\} = (H + a)^2 + d, \quad \{P, P^\dagger\} = H. \tag{54}$$

The compatibility of two supersymmetries is achieved by the following constraint $\chi = 2f$ and by the nonlinear second-order differential equation

$$f^2 + \frac{f''}{2f} - \left(\frac{f'}{2f}\right)^2 - \frac{d}{4f^2} - a = \chi^2 = 4f^2. \tag{55}$$

with solutions parameterized by two integration constants. Therefore the existence of two SUSY reduces substantially the class of potentials for which they may appear. Evidently Eq. (55) can be integrated to,

$$(f')^2 = 4f^4 + 4af^2 + 4G_0 f - d \equiv \Phi_4(f), \tag{56}$$

where G_0 is a real constant.

The solutions of this equation are elliptic functions which can be easily found in the implicit form,

$$\int_{f_0}^{f(x)} \frac{df}{\sqrt{\Phi_4(f)}} = \pm(x - x_0), \tag{57}$$

where f_0 and x_0 are real constants.

It can be shown that they are nonsingular if:
a) $\Phi_4(f)$ has three different real roots and the double root $\beta/2$ is either the maximal one or a minimal one,

$$\Phi_4(f) = 4(f - \frac{\beta}{2})^2 \left((f + \frac{\beta}{2})^2 - (\beta^2 - \epsilon) \right), \quad 0 < \epsilon < \beta^2. \tag{58}$$

Then there exits a relation between constants a, d, G_0 in terms of coefficients β, ϵ,

$$a = \epsilon - \frac{3\beta^2}{2} < 0, \quad G_0 = \beta(\beta^2 - \epsilon), \quad d = \beta^2 \left(\frac{3\beta^2}{4} - \epsilon \right). \tag{59}$$

The constant f_0 is taken between the double root and a nearest simple root.
b) $\Phi_4(f)$ has two different real double roots which corresponds in (58), (59) to $G_0 = 0, \quad \beta^2 = \epsilon > 0, \quad a = -\epsilon/2, \quad d = -\epsilon^2/4$. The constant f_0 ranges between the roots.

The corresponding potentials $V_{1,2}$ are well known [117] to be reflectionless, with one bound state at the energy $(\beta^2 - \epsilon)$ and with the continuum spectrum starting from β^2. Respectively the scattering wave function is proportional to $\exp(ikx)$ with $k = \sqrt{E - \beta^2}$.

In the case a) the potentials coincide in their form and differ only by shift in the coordinate ("Darboux Displacement" [83]),

$$V_{1,2} = \beta^2 - \frac{2\epsilon}{\mathrm{ch}^2 \left(\sqrt{\epsilon}(x - x_0^{(1,2)}) \right)}, \quad x^{(1,2)} = x_0 \pm \frac{1}{4\sqrt{\epsilon}} \ln \frac{\beta - \sqrt{\epsilon}}{\beta + \sqrt{\epsilon}}, \tag{60}$$

and in the case b) one of the potentials can be chosen constant (being a limit of infinite displacement),

$$V_1 = \beta^2, \quad V_2 = \beta^2 \left(1 - \frac{2}{\mathrm{ch}^2 \left(\beta(x - x_0) \right)} \right), \tag{61}$$

For these potentials one can elaborate extended SUSY algebra.

The initial algebra is given by the relations (54). It must be completed by the mixed anti-commutators

$$\{K, P^\dagger\} = \{K^\dagger, P\}^\dagger = \mathcal{B}(H) - i\mathcal{E}(H), \tag{62}$$

where the first term is (see the next section) a polynomial of the Super-Hamiltonian and the second one is in general not. In our case the first, polynomial symmetry operator turns out to be constant, $\mathcal{B}(H) = G_0$ when taking into account (52) and (56). Meanwhile the second operator reads,

$$\mathcal{E}(H) = i \left[\mathbf{I}\, \partial^3 - \left(a\mathbf{I} + \frac{3}{2}\mathbf{V}(x) \right) \partial - \frac{3}{4}\mathbf{V}'(x) \right], \tag{63}$$

in the notations $H \equiv -\partial^2 \mathbf{I} + \mathbf{V}(x)$. By construction the operator $\mathcal{E}(H)$ realizes a new symmetry for the Super-Hamiltonian. Directly from Eq. (63) one derives that,

$$\mathcal{E}^2(H) = H\left[(H+a)^2 + d\right] - G_0^2 = (H-E_b)^2(H-\beta^2), \tag{64}$$

where $E_b = \beta^2 - \epsilon$ is the energy of a bound state. Thus (some of) the zero modes of $\mathcal{E}(H)$ characterize either bound states or zero-energy states in the continuum. However there exist also the non-normalizable, unphysical zero-modes corresponding to $E = E_b, \beta^2$. We remark that in the case b) only the Hamiltonian h^- has a bound state. Hence the physical zero modes of $\mathcal{E}(H)$ may be either degenerate [case a), broken SUSY] or non-degenerate [case b), unbroken SUSY].

The square root of (64) can be established unambiguously from the analysis of transmission coefficients,

$$\mathcal{E}(H) = (H-E_b)\sqrt{H-\beta^2}. \tag{65}$$

We notice that the symmetry operator (63), (65) is irreducible, *i.e.* the binomial $(H - E_b)$ cannot be "stripped off". Indeed the elimination of this binomial would convert the third-order differential operator (63) into an essentially nonlocal operator. The sign of square root in (65) is fixed from the conventional asymptotics of scattering wave functions $\sim \exp(ikx)$ and the asymptotics $V_{1,2} \longrightarrow \beta^2$ by comparison of this relation with Eq. (63).

When taking Eq. (65) into account one finds the mixed anti-commutators of the extended SUSY algebra (62) in a non-polynomial form,

$$\{K,\, P^\dagger\} = \{K^\dagger,\, P\}^\dagger = G_0 - i(H-E_b)\sqrt{H-\beta^2}. \tag{66}$$

Thus the "central charge" of this extended SUSY is built of the elements (54) and (66) and evidently cannot be diagonalized by a unitary rotation with elements polynomial in H. Therefore the algebra must be considered to be extended in the class of differential operators of finite order.

Let us now clarify the non-uniqueness of the higher-order supercharge and its role in the classification of the Polynomial SUSY. For arbitrary ζ in (47) one obtains

$$\begin{aligned} &\{K^{(\zeta)},\, \left(K^{(\zeta)}\right)^\dagger\} = H^2 + (2a+\zeta^2)H + a^2 + d + 2\zeta G_0 = (H+a_\zeta)^2 + d_\zeta, \\ &a_\zeta = a + \frac{1}{2}\zeta^2, \quad d_\zeta = d + 2\zeta G_0 - a\zeta^2 - \frac{1}{4}\zeta^4 \equiv -\Phi_4(-\frac{\zeta}{2}), \end{aligned} \tag{67}$$

where $\Phi_4(f)$ is defined in Eq. (56).

For the extended SUSY one can discover that the previous classification (Section 3) of irreducible ladders (Darboux transformations) may fail. Indeed, the sign of d_ζ, in general, depends on the choice of ζ. For instance, let us consider the case a) when

$$d_\zeta = -\frac{1}{4}(\zeta+\beta)^2\left[(\zeta-\beta)^2 - 4(\beta^2-\epsilon)\right]. \tag{68}$$

Evidently if ζ lies between the real roots of the last factor in (68) then d_ζ is positive and otherwise it is negative. But two real roots always exist because $\beta^2 > \epsilon$. Thereby the sign of d_ζ can be made negative as well as positive without any change in the Super-Hamiltonian. Hence in the case when the Polynomial SUSY is an extended one, with two sets of supercharges, the irreducibility of type I of a Polynomial SUSY algebra does not signify any invariant characteristic of potentials.

7. Non-stationary Schrödinger equations: intertwining and hidden symmetry

Discussions of symmetry properties of the time dependent Schrödinger equation have a long history, see for instance [124] and references quoted therein as well as [125]. In these discussions the potentials concerned are mainly time independent, see e.g. [118], [120]. Here our aim is to elucidate that many of the nonlinear SUSY constructions illustrated before can be implemented also in the Schrödinger time dependent framework [97], [119], [121].

Certainly, general first- and higher-order intertwining relations between non-stationary one-dimensional Schrödinger operators can be easily introduced. But already in the first-order case the intertwining relations imply some hidden symmetry which in turn leads to a specific quantum dynamics when the evolution is described by quantum orbits and results in the R-separation of variables[122]. Second-order intertwining operators [122],[123] and the corresponding non-linear SUSY give rise to the quantum motion governed by the spectrum generating algebras.

Let us start with intertwining relations of the *non-stationary Schrödinger operator*

$$\mathcal{S}[V] = i\partial_t + \partial_x^2 - V(x,t) \ . \tag{69}$$

Here $\partial_t = \partial/\partial t$ and $\partial_x = \partial/\partial x$ denote the partial derivatives with respect to time and position : we will denote these derivatives, if applied to some function f, by a dot and prime, respectively. Hence, we use the notation $\dot{f}(x,t) = (\partial_t f)(x,t)$ and $f'(x,t) = (\partial_x f)(x,t)$.

The most general intertwining operator of first order [122] is given by

$$q_t^+ = \xi_0(x,t)\partial_t + \xi_1(x,t)\partial_x + \xi_2(x,t) \tag{70}$$

with, in general, complex-valued functions ξ_0, ξ_1 and ξ_2.The possibility of a complexification of the intertwining (Darboux) (first and also higher order) operator was emphasized by [123]. Note also that in contrast to [119] the formalism of [122] allow a priori for a first-order operator in ∂_t.

For the above defined Schrödinger operator (69) the intertwining relation reads

$$\mathcal{S}[V_1]q_t^+ = q_t^+\mathcal{S}[V_2] \ , \tag{71}$$

where the functions ξ_i $(i = 0,1,2)$ and $V_{1,2}$ are not independent. It can be also represented in the SUSY form Eq. (5) when the stationary Hamiltonians $h^\pm$ are extended to the Schrödinger operators $\mathcal{S}[V_{1,2}]$, then

$$[\hat{\mathcal{S}}_t, Q_t] = 0, \quad \hat{\mathcal{S}}_t = \begin{pmatrix} \mathcal{S}[V_1] & 0 \\ 0 & \mathcal{S}[V_2] \end{pmatrix}, \quad Q_t = \begin{pmatrix} 0 & q_t^+ \\ 0 & 0 \end{pmatrix}. \tag{72}$$

Inserting the explicit forms of the Schrödinger operators (69) and the intertwining operator (70) into relation (71) it was found [122] that ξ_0 and ξ_1 may depend only on time, i.e. $\xi_0' = 0 = \xi_1'$. The assumption that ξ_0 does not vanish identically then leads to the consequence that also the potential difference $V_1 - V_2$ does depend only on time. This is a rather uninteresting case and, therefore, we set $\xi_0 \equiv 0$ without a loss of generality. Making now the following choice of appropriate variables $\xi_1(t) = e^{i\beta(t)}\rho(t)$ and $\xi_2(x,t) = e^{i\beta(t)}\rho(t)\omega'(x,t)$ with real β, positive ρ and complex ω functions one finds

$$\begin{aligned} V_1(x,t) &= \omega'^{\,2}(x,t) + \omega''(x,t) - i\dot\omega(x,t) + \alpha(t) - \dot\beta(t) + i\dot\rho(t)/\rho(t)\ , \\ V_2(x,t) &= \omega'^{\,2}(x,t) - \omega''(x,t) - i\dot\omega(x,t) + \alpha(t)\ , \end{aligned} \tag{73}$$

where α is some time-dependent complex-valued integration constant. Again one may set [122] $\beta \equiv 0$ without loss of generality. Furthermore, one may also take $\alpha \equiv 0$ because it can always be absorbed in ω by the shift $\omega \to \omega - i\int dt\,\alpha$. Hence, we are left with

$$\begin{aligned} V_1(x,t) &= \omega'^{\,2}(x,t) + \omega''(x,t) - i\dot\omega(x,t) + i\dot\rho(t)/\rho(t)\ , \\ V_2(x,t) &= \omega'^{\,2}(x,t) - \omega''(x,t) - i\dot\omega(x,t)\ . \end{aligned} \tag{74}$$

Here the so-called super-potential ω is still not arbitrary as the potentials are to be real. This can, for example, be achieved by assuming a stationary real super-potential. However it leads to the standard stationary SUSY QM discussed previously. Alternatively, we will consider a complex super-potential

$$\omega(x,t) = \omega_R(x,t) + i\omega_I(x,t) \tag{75}$$

with real functions ω_R and ω_I. The reality condition Im V_1 = Im $V_2 = 0$ implies

$$2(\omega_I)'' + \dot\rho/\rho = 0\ , \qquad 2(\omega_R)'(\omega_I)' - (\omega_I)'' - \dot\omega_R = 0\ , \tag{76}$$

which can easily be integrated to

$$\begin{aligned} \omega_I(x,t) &= -\frac{1}{4}\,\frac{\dot\rho(t)}{\rho(t)}\,x^2 + \frac{1}{2}\,\rho(t)\dot\mu(t)x + \gamma(t)\ , \\ \omega_R(x,t) &= \frac{1}{2}\,\ln\rho(t) + K\Big(x/\rho(t) + \mu(t)\Big)\ , \end{aligned} \tag{77}$$

where μ and γ are arbitrary real functions of time and K is an arbitrary real function of the variable $y = x/\rho + \mu$. In terms of these functions the final form of the two partner potentials is

$$\begin{aligned} V_{1,2}(x,t) = {}& \frac{1}{\rho^2(t)}\left[K'^2(y) \pm K''(y)\right] \\ & -\frac{\ddot\rho(t)}{4\rho(t)}\,x^2 + \left(\dot\rho(t)\dot\mu(t) + \frac{\rho(t)\ddot\mu(t)}{2}\right)x - \frac{\rho^2(t)\dot\mu^2(t)}{4} + \dot\gamma(t) \end{aligned} \tag{78}$$

and the intertwining operator reads

$$q_t^+(x,t) = \rho(t)\partial_x + K'\Big(x/\rho(t) + \mu(t)\Big) - \frac{i}{2}\left(\dot\rho(t)x - \rho^2(t)\dot\mu(t)\right). \tag{79}$$

Let us demonstrate [122] that the non-stationary Schrödinger equation $\mathcal{S}[V_{1,2}]\psi_{1,2} = 0$ with potentials given in Eq. (78) (which is equivalent to the intertwining (71)) admits a separation of variables. In fact, after the transformation

$$y = x/\rho(t) + \mu(t)\ , \qquad \psi_{1,2}(x,t) = \frac{1}{\sqrt{\rho(t)}}\, e^{-i\omega_I(x,t)}\phi_{1,2}(y,t) \equiv \Omega(x,t)\phi_{1,2}(y,t) \tag{80}$$

this Schrödinger equation becomes quasi-stationary [126]

$$i\rho^2(t)\partial_t\phi_{1,2}(y,t) = \left[-\partial_y^2 + K'^2(y) \pm K''(y)\right]\phi_{1,2}(y,t)\ , \tag{81}$$

which is obviously separable in y and t. Hence, the solutions of the original Schrödinger equations have the general form $\psi(x,t) = \Omega(y,t)Y(y)T(t)$ which is known as the R-separation of variables [127]. In other words, for any pair of Schrödinger operators $\mathcal{S}[V_{1,2}]$, which admits a first-order intertwining relation (71) there exists a transformation (80) to some new coordinate in which the potentials become stationary (see also [121]). Notice that the transformation associated with the special case $\rho(t) = 1$ and $\mu(t) = vt$ with constant velocity v corresponds to the Galileo transformation. See, for example, the textbook [128].

This R-separation of variables is certainly related to the existence of a symmetry operator. First, one can directly verify the adjoint intertwining relation for real potentials,

$$q_t^-\mathcal{S}[V_1] = \mathcal{S}[V_2]q_t^- \tag{82}$$

where

$$q_t^- \equiv (q_t^+)^\dagger = -\rho(t)\partial_x + K'\Big(x/\rho(t) + \mu(t)\Big) + \frac{i}{2}\Big(\dot\rho(t)x - \rho^2(t)\dot\mu(t)\Big)\ . \tag{83}$$

Then from (71), (72) and (82) we obtain the closure of the SUSY algebra,

$$\left\{Q_t, Q_t^\dagger\right\} = \mathcal{R}_t, \quad \left[\hat{\mathcal{S}}_t, \mathcal{R}_t\right] = 0\ , \quad Q_t^\dagger = (Q_t)^\dagger\ , \tag{84}$$

where the symmetry operator $\mathcal{R}_t$ has the following components

$$\begin{aligned} R_t^\pm = q_t^\pm q_t^\mp &= -\rho^2(t)\partial_x^2 + \frac{i}{2}\Big(\rho\dot\rho(t)\{x,\partial_x\} - 2\rho^3(t)\dot\mu(t)\partial_x\Big) \\ &\quad + \Big[K'\Big(x/\rho(t) + \mu(t)\Big)\Big]^2 \pm K''\Big(x/\rho(t) + \mu(t)\Big) + \frac{1}{4}\Big(\dot\rho(t)x - \rho^2(t)\dot\mu(t)\Big)^2 \\ &= \exp\{-i\omega_I(x,t)\}\left[-\partial_y^2 + K'^2(y) \pm K''(y)\right]\exp\{i\omega_I(x,t)\}. \end{aligned} \tag{85}$$

Thus the quasi-stationary Hamiltonians in Eq. (81) are just unitary equivalent to the symmetry operators $R_t^\pm$. It means that the supersymmetry entails the separation of variables because it provides a new symmetry. As a consequence the quantum dynamics splits in orbits with a given eigenvalue of the symmetry operator.

8. Second-order intertwining for stationary potentials: symmetry operators and spectrum generating algebra

Now we will report on the intertwining of a pair of Schrödinger operators $\mathcal{S}[V_1]$ and $\mathcal{S}[V_2]$ by second-order (intertwining) operators of the form:

$$q_t^+(x,t) = G(x,t)\partial_x^2 - 2F(x,t)\partial_x + B(x,t)\ . \tag{86}$$

We will explore the connection of the time-dependent SUSY charges with appearance of the spectrum generating (oscillator like) algebras for the corresponding Hamiltonians.

As in the first-order case it can be shown [122] that the inclusion of an additional term being of first order in ∂_t leads to the trivial situations where the difference $V_1 - V_2$ depends on the time t only. Furthermore, from the intertwining relation (71) with above q_t^+ one can conclude that the function G may not depend on x and similarly to the discussion in the previous section it is even possible to exclude a phase. In other words, without loss of generality $G(x,t) \equiv g(t)$ and consider from now on an intertwining operator of the form

$$q_t^+(x,t) = g(t)\partial_x^2 - 2F(x,t)\partial_x + B(x,t) \; . \tag{87}$$

In [122] particular solutions of the intertwining relation (71) were constructed with q_t^+ as given above . In this section we shall analyze the solutions of the intertwining relation (71) for the case where both potentials V_1 and V_2 are stationary, *i.e.* do not depend on t.

One class of such solutions is known from [54]. Assuming a supercharge q_t^+ with real coefficient functions independent on t, one finds that the corresponding solutions of (71) coincide with those of the stationary intertwining relations $(-\partial_x^2 + V_1(x))q^+(x) = q^+(x)(-\partial_x^2 + V_2(x))$ from [54].

Here we are interested in more general solutions of (71) when operators q_t^+ depend on t,

$$(i\partial_t - h^+)q_t^+(x,t) = q_t^+(x,t)(i\partial_t - h^-) \; , \tag{88}$$

with standard stationary Hamiltonians $h^\pm = -\partial_x^2 + V_{1,2}(x)$ but explicitly time-dependent intertwining operators.

Let us employ the suitable ansatz with simple t-dependence in (87),

$$q_t^+(x,t) = M^+(x) + A(t)a^+(x) \; , \tag{89}$$

where

$$M^+(x) \equiv \partial_x^2 - 2f(x)\partial_x + b(x) \; , \qquad a^+(x) \equiv \partial_x + W(x) \; . \tag{90}$$

Here all functions besides A are considered to be real. We also assume $A \not\equiv 0$. With this ansatz the intertwining relation (88) can be shown [122] to yield

$$\begin{aligned} &i\dot{A} = 2\,\tilde{m} + 2\,m\,A \; , \\ &h^+M^+ - M^+h^- = 2\,\tilde{m}\,a^+ \; , \\ &h^+a^+ - a^+h^- = 2\,m\,a^+ \; , \end{aligned} \tag{91}$$

with real constants $\tilde{m}$ and m.

We find it interesting to focus on the case $m \neq 0$ to explore certain spectrum generating algebras. The first equation in (91) immediately leads to

$$A(t) = m_0 e^{-2imt} - \tilde{m}/m \tag{92}$$

with a real m_0, and

$$q_t^+(x,t) = \partial_x^2 - \left(2f(x) + \frac{\tilde{m}}{m}\right)\partial_x + b(x) - \frac{\tilde{m}}{m}W(x) + m_0 e^{-2imt}a^+(x)\ . \quad (93)$$

It is obvious that without loss of generality we may set $\tilde{m} = 0$ as a non-vanishing $\tilde{m}$ may always be absorbed via a proper redefinition of f and b, *i.e.* of the operator M.

As a consequence, the second relation in (91) leads to a second-order intertwining between h^+ and h^-. This has already been considered in [54] and it was found that the potentials V_1, V_2 and the function b can be expressed in terms of f and two arbitrary real constants a and d:

$$\begin{aligned} V_{1,2}(x) &= \mp 2f'(x) + f^2(x) + \frac{f''(x)}{2f(x)} - \frac{f'^2(x)}{4f^2(x)} - \frac{d}{4f^2(x)} - a\ , \\ b(x) &= -f'(x) + f^2(x) - \frac{f''(x)}{2f(x)} + \frac{f'^2(x)}{4f^2(x)} + \frac{d}{4f^2(x)}\ . \end{aligned} \quad (94)$$

The corresponding second-order SUSY algebra generated by the supercharge M is similar to (54),

$$\{M,\, M^\dagger\} = (H+a)^2 + d \equiv \mathcal{P}_2(H). \quad (95)$$

Note that cases with $d < 0$ are reducible ones.

One may find some similarities between the present intertwining algebra (91) and the extended SUSY relations discussed in Section 5. But we emphasize that now for $m \neq 0$ the last relation in (91) does not generate a second SUSY. Rather it creates the equivalence of relatively shifted spectra of two Hamiltonians h^+ and h^- which is typical for spectrum generating algebras. Specifically

$$a^+a^- = h^+ - m + c; \quad a^-a^+ = h^- + m + c, \quad (96)$$

where c is a real constant. Therefore the reflectionless potentials found in Section 5 are produced only in the limit of $m = 0$. For this reason we use here the notations for relevant operators different from those ones in Section 5.

The genuine spectrum generating algebra for stationary Hamiltonians $h^\pm$ can be derived from Eq. (91)

$$\begin{aligned} &[h^+, G_+] = -2mG_+\ , \quad G_+ \equiv M^+a^-\ , \\ &[h^+, G_+^\dagger] = 2mG_+^\dagger\ , \quad G_+^\dagger \equiv a^+M^-\ , \\ &[h^-, G_-^\dagger] = 2mG_-^\dagger\ , \quad G_-^\dagger \equiv M^-a^+\ , \\ &[h^-, G_-] = -2mG_-\ , \quad G_- \equiv a^-M^+\ , \end{aligned} \quad (97)$$

where $a^- = (a^+)^\dagger$ and $M^- = (M^+)^\dagger$. The closure of this spectrum generating algebra is a polynomial deformation of Heisenberg algebra [129],

$$[G_\pm^\dagger, G_\pm] = F^\pm(h^\pm). \quad (98)$$

The explicit form of the polynomials $F^\pm(x)$ can be obtained with the help of Eqs. (91) and (96) for $\tilde{m} = 0$. For instance,

$$\begin{aligned} &G_+^\dagger G_+ = (h^+ - m + c)\,\mathcal{P}_2(h^+ - 2m); \\ &G_+G_+^\dagger = (h^+ + m + c)\,\mathcal{P}_2(h^+), \end{aligned} \quad (99)$$

where the notations from Eqs. (95) and (96) are employed. The polynomials $F^{\pm}(x)$ turn out to be different for the isospectral partners $h^{\pm}$,

$$F^{+}(h^{+}) = -6m(h^{+})^2 + 4m(2m-2a-c)h^{+} - 2m[a^2+d+2(a-m)(c-m)];$$
$$F^{-}(h^{-}) = -6m(h^{-})^2 - 4m(2m+2a+c)h^{-} - 2m[a^2+d+2(a+m)(c+m)]. \qquad (100)$$

Hence the two spectrum generating algebras are, in general, different that is essentially due to the shift in intertwining relations (96). There is a formal discrete symmetry between their constants and Hamiltonians $h^{+}, a, c \Longrightarrow -h^{-}, -a, -c$.

The intertwining relation (88) and its adjoint give rise to the symmetry operators $q_t^{+} q_t^{-}$ and $q_t^{-} q_t^{+}$ for $(i\partial_t - h^{+})$ and $(i\partial_t - h^{-})$, respectively. Using Eqs. (92),(95),(96) and after elimination of polynomials of the Hamiltonians $h^{\pm}$ these symmetry operators may be reduced to the form

$$\begin{aligned} R^{+}(x,t) &= m_0\left[e^{2imt}G_{+} + e^{-2imt}G_{+}^{\dagger}\right], \\ R^{-}(x,t) &= m_0\left[e^{2imt}G_{-} + e^{-2imt}G_{-}^{\dagger}\right]. \end{aligned} \qquad (101)$$

As our potentials do not depend on time the operators $R^{\pm}(x, t+\Delta)$ with a time shift Δ are also symmetry operators for the same Schrödinger equation,

$$\left[R^{\pm}_{(t+\Delta)}, S_t\right] = 0. \qquad (102)$$

In particular, time derivatives $\dot{R}^{\pm}(x,t)$ of hermitian symmetry operators $R^{\pm}(x,t)$ form an independent set of hermitian symmetry operators which do not commute between themselves. Similar results have also been obtained in [118] using a different approach. We see that the non-stationary SUSY delivers the symmetry operators which encode the entire set of spectrum generating algebras

$$e^{2imt}G_{\pm} = \frac{1}{2m_0}R^{\pm}(x,t) - \frac{i}{4mm_0}\dot{R}^{\pm}(x,t). \qquad (103)$$

The natural question concerns the reducibility of the second-order intertwining operator (87) to a pair of consecutive first-order operators. Progress in classification of possible irreducible transformations has been made in [122] though more work must be done toward the full classification.

9. Conclusions and perspectives

The purpose of this short review has been two-fold: to elucidate the recent progress in Nonlinear SUSY realization for a broad community of spectral designers and to draw reader's attention to a variety of SUSY extensions which yield new QES potential systems and illuminate some old ones. With the experience from the previous sections the general SUSY QM can be thought of as governed by the extended nonlinear SUSY algebra with $\mathcal{N}$ pairs of nilpotent supercharges Q_j, $Q_j^{\dagger}$ and a number of hermitian hidden-symmetry differential operators $R_\alpha = R_\alpha^{\dagger}, \quad [R_\alpha, R_\beta] = 0; \quad 0 \le \alpha, \beta \le M$. Such a SUSY algebra takes the modified form,

$$\begin{aligned} &[R_\alpha, Q_k] = \left[R_\alpha, Q_k^{\dagger}\right] = 0; \quad \{Q_j, Q_k\} = \left\{Q_j^{\dagger}, Q_k^{\dagger}\right\} = 0; \\ &\left\{Q_j, Q_k^{\dagger}\right\} = \mathcal{P}(R_\alpha); \end{aligned} \qquad (104)$$

We notice that, first, the Super-Hamiltonian itself is included into the set of symmetry operators, say for $\alpha = 0$, $R_0 \equiv H$ and, second, not all the symmetry operators are necessarily present in the algebraic closure (104) (see Sections 7,8).

On the other hand, there remains a plenty of open questions and challenges to be solved.

- In the first half of this paper we have given a systematic analysis of reducibility vs. irreducibility of type I, II, III in the one-dimensional QM. However the higher-order irreducibility needs more efforts to prove the exhaustive completeness of the classification in Section 3.
- For SUSY extensions it may be of interest to find pairs of (quasi)isospectral potentials admitting hidden symmetries which are related by a type-II irreducible Darboux transformation.
- The irreducibility classification for non-stationary potentials as well as the existence of extended SUSY is very welcome to be investigated and new applications to be found, in particular, to explore spectrum generating algebras (see Section 8).
- The similarity of the Schrödinger equation to the Fokker-Planck one allows [96, 122, 130] to find the SUSY scheme for generating new solutions of the latter equation. One can be tempted to develop a more exhaustive analysis of how to produce SUSY clones using the ideas of the conventional nonlinear SUSY outlined here. But attention should be focused on the non-hermiticity of the Fokker-Planck operator and on the fact that the equivalent of the wave function is a positive and properly normalized probability function.
- Matrix (coupled channel) systems represent a rich and not fully scanned field of extended SUSY systems with hidden symmetries. While certain interesting matrix potentials have been explored [45], [131]-[134] it is clear that in this case the way to a comprehensive understanding of irreducible building blocks for spectral design is still long.
- The polynomial SUSY in two dimensions has already brought a number of examples of new type of irreducible SUSY with hidden symmetries of higher-order in derivatives [55, 57, 135]. One may expect a variety of new types of irreducible SUSY for third-order (and higher-order) supercharges as well as new discoveries in three dimensions.
- Complex potentials ([136]–[142]) seem to offer less problematic generalizations of QM with hermitian Hamiltonians as compared to matrix and multi-dimensional QM. Therefore many of tools and results of one-dimensional SUSY QM are expected to be applicable when a potential is complex [85]. However, as it was recently remarked in [141] there exist non-hermitian Hamiltonians which are not diagonalizable but at best can be reduced to a Jordan form. For the latter ones special care must be taken to derive the isospectrality and to build SUSY ladders.

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